Tropical Cyclone
Axisymmetric Physics

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Program

- Brief Overview
- Steady-state energetics and physics
- Structure
- Intensification physics
Overview: What is a Tropical Cyclone?

A *tropical cyclone* is a nearly symmetric, warm-core cyclone powered by wind-induced enthalpy fluxes from the sea surface.
Global Climatology

Tracks of all tropical cyclones in the historical record from 1851 to 2010. The tracks are colored according to the maximum wind at 10 m altitude, on the scale at lower right.
The View from Space
View of the eye of Hurricane Katrina on August 28th, 2005, as seen from a NOAA WP-3D hurricane reconnaissance aircraft.
Hurricane Structure: Wind Speed

Azimuthal component of wind

< 11 5 ms\(^{-1}\) - > 60 ms\(^{-1}\)
Vertical Air Motion

Updraft Speed

Strong upward motion in the eyewall
Absolute angular momentum per unit mass

\[ M = rV + \Omega r^2 \]
Physics of Mature Hurricanes

References:

Emanuel, *J. Atmos. Sci.*, 1986

Rousseau-Rizzi & Emanuel, *J. Atmos. Sci.*, 2019 (in early online release)
Cross-section through a Hurricane & Energy Production

- Nearly isothermal expansion
- Isothermal compression
- Adiabatic expansion
- Adiabatic compression
- Nearly isothermal expansion

At point B:
- \( T = 27°C = 300K \)

Points:
- Eye
- Ocean surface
- Low entropy
- Radius (km)
- Height above ocean
Carnot Theorem: Maximum efficiency results from a particular energy cycle:

- Isothermal expansion
- Adiabatic expansion
- Isothermal compression
- Adiabatic compression

Note: Last leg is not adiabatic in hurricanes: Air cools radiatively. But since the environmental temperature profile is moist adiabatic, the amount of radiative cooling is the same as if air were saturated and descending moist adiabatically.

Maximum rate of energy production:

\[ P = \frac{T_s - T_o}{T_s} \dot{Q} \]
Total rate of heat input to hurricane:

\[
\dot{Q} = 2\pi \int_0^{r_0} \rho \left[ C_k \| \mathbf{V} \| \left( k_0^* - k \right) + C_D \| \mathbf{V} \|^3 \right] r dr
\]

In steady state, energy production is used to balance frictional dissipative heating:

\[
D = 2\pi \int_0^{r_0} \rho \left[ C_D \| \mathbf{V} \|^3 \right] r dr
\]
Differential Carnot Cycle
\[ D = \frac{T_s - T_o}{T_s} \dot{Q} \]

\[
\rho \left[ C_D \left| V_{max} \right|^3 \right] = \frac{T_s - T_o}{T_s} \rho \left[ C_k \left| V_{max} \right| \left( k_0^* - k \right) + C_D \left| V_{max} \right|^3 \right]
\]

\[
\rightarrow \rho \left[ C_D \left| V_{max} \right|^3 \right] = \frac{T_s - T_o}{T_o} \rho \left[ C_k \left| V_{max} \right| \left( k_0^* - k \right) \right]
\]

\[
\rightarrow \left| V_{max} \right|^2 \approx \frac{C_k}{C_D} \frac{T_s - T_o}{T_o} \left( k_0^* - k \right)
\]

Note that this is valid between ANY two streamlines in the region of ascent.
Simulations with cloud-permitting axisymmetric model for different horizontal mixing lengths

Rousseau-Rizzi and Emanuel, *J. Atmos. Sci.*, 2019
Derivation of gradient wind potential intensity from thermal wind balance

\[ M_g = rV_g + \frac{1}{2} f r^2 \]

\[ \frac{\partial \phi}{\partial r} = \frac{V_g^2}{r} + fV = \frac{M_g^2}{r^3} - \frac{1}{4} f^2 r \]

Gradient balance

\[ \frac{\partial \phi}{\partial p} = -\alpha \]

Hydrostatic balance

\[ \frac{2M_g}{r^3} \frac{\partial M_g}{\partial p} = - \frac{\partial \alpha}{\partial r} = - \left( \frac{\partial \alpha}{\partial s^*} \right)_p \frac{\partial s^*}{\partial r} \]

Thermal wind

\[ \frac{2M_g}{r^3} \frac{\partial M_g}{\partial p} = - \left( \frac{\partial T}{\partial p} \right)_{s^*} dM_g \frac{\partial M_g}{\partial r} \]
\[
\frac{2M_g}{r^3} \frac{\partial M_g}{\partial p} = - \left( \frac{\partial T}{\partial p} \right)_{s^*} \frac{ds^*}{dM_g} \frac{\partial M_g}{\partial r}
\]

\[
\rightarrow \frac{1}{r^3} \left( \frac{\partial r}{\partial p} \right)_{M_g} = \frac{1}{2M_g} \frac{ds^*}{dM_g} \left( \frac{\partial T}{\partial p} \right)_{s^*}
\]

Integrate in pressure:

\[
\frac{M_g}{r_b^2} - \frac{M_g}{r_o^2} = - \left( T_b - T_o \right) \frac{ds^*}{dM_g}
\]

\[
\rightarrow \frac{V_{gb}}{r_b} = \frac{V_{go}}{r_o} - \left( T_b - T_o \right) \frac{ds^*}{dM_g}
\]  \(\text{(1)}\)
Define outflow to be where \( V_o = 0 \)

\[
V_{gb} = -r_b \left( T_b - T_o \right) \frac{d s^*}{d M_g}
\]

Convective criticality: \( s^* = s_b \)

\[
\rightarrow V_{gb} = -r_b \left( T_b - T_o \right) \frac{d s_b}{d M_g}
\]

(1)
$ds_b/dM_g$ determined by boundary layer processes
Put (1) in differential form:

\[(T_b - T_o) \frac{ds}{dt} + \frac{M_g}{r^2} \frac{dM_g}{dt} = 0.\]  \hspace{1cm} (2)

Integrate entropy equation through depth of boundary layer:

\[h \frac{d\bar{S}}{dt} = \frac{1}{T_s} \left[ C_k |\mathbf{V}| (k_0^* - k) + C_D |\mathbf{V}|^3 \right] \hspace{1cm} (3)\]

Integrate angular momentum equation through depth of boundary layer:

\[h \frac{d\bar{M}}{dt} \approx h \frac{dM_g}{dt} = -C_D rV |\mathbf{V}| \hspace{1cm} (4)\]
Substitute (3) and (4) into (2) and equate $V$ with $|V|$:

$$\rightarrow |V|^2 = \frac{C_k}{C_D} \frac{T_s - T_o}{T_o} \left( k_0^* - k \right)$$

(5)

Same answer as from Carnot cycle. This is still not a closed expression, since we have not determined the boundary layer enthalpy, $k$ or the outflow temperature, $T_o$. 
What Determines Outflow Temperature?

Simulations with Cloud-Permitting, Axisymmetric Model
Saturation entropy (contoured) and $V=0$ line (yellow)
Streamfunction (black contours), absolute temperature (shading) and V=0 contour (white)
Angular momentum surfaces plotted in the V-T plane. Red curve shows shape of balanced M surface originating at radius of maximum winds. Dashed red line is ambient tropopause temperature.
Richardson Number (capped at 3). Box shows area used for scatter plot.
Vertical Diffusivity (m²s⁻¹)
Implications for Outflow Temperature

\[ Ri = \frac{\Gamma_d \frac{\partial S_d}{\partial z}}{\left( \left| \frac{\partial V}{\partial z} \right| \right)^2} \approx \frac{r^2 \Gamma_m \frac{d s^*}{d M}}{\frac{\partial M}{\partial z}}. \]

\[ \rightarrow \frac{\partial M}{\partial z} \approx \frac{r^2 \Gamma_m \frac{d s^*}{d M}}{R_i_c}. \]

\[ \frac{\partial s^*}{\partial z} = \frac{d s^*}{d M} \frac{\partial M}{\partial z}, \]
But the vertical gradient of saturation entropy is related to the vertical gradient of temperature:

\[ \frac{\partial s^*}{\partial z} \approx \frac{r_t^2 \Gamma_m \left( \frac{ds^*}{dM} \right)^2}{Ri_c}. \] (5)

Use definition of \( s^* \) and C.-C.:

\[ \left( \frac{\partial T}{\partial s^*} \right)_p = \left( \frac{\partial T}{\partial s^*} \right)_p + \left( \frac{\partial T}{\partial p} \right)_{s^*}, \] (6)

\[ \left( \frac{\partial T}{\partial s^*} \right)_p = \frac{T \frac{\partial p}{\partial T}}{c_p} \left[ 1 + \frac{L_v q^*}{R_v c_p T^2} \right]. \] (7)
Substitute (16) into (15) and use hydrostatic equation:

\[
\frac{\partial T}{\partial s^*} = \frac{T}{c_p} - \frac{\Gamma_m}{\partial z} \left[ 1 + \frac{L_v q^*}{R_v c_p T^2} \right].
\]  

If \( V_b^2 << c_p \frac{(T_b - T_o)^2}{T_b} \left( 1 + \frac{L_v q^*}{R_v c_p T^2} \right) \left[ R_i c \frac{r_b^2}{r_t^2} \right] \) we can neglect first term on left of (8)

Substitute (5) into (8):

\[
\frac{\partial T_o}{\partial s^*} \approx - \frac{R_i c}{r_t^2} \left( \frac{dM}{ds^*} \right)^2.
\]  

Gives dependence of Outflow T on s*
Using
\[ \frac{\partial T}{\partial s^*} = \frac{\partial T}{\partial M} \frac{dM}{ds^*}, \]

We can re-write (9) as
\[ \frac{\partial T_o}{\partial M} \equiv -\frac{Ri_c}{r_t^2} \left( \frac{dM}{ds^*} \right). \] (10)

We can also re-write (1)
\[ M_b = r_b^2 \left( \frac{1}{2} f - (T_b - T_o) \frac{ds^*}{dM} \right) \] (11)
as
\[ h \frac{ds_b}{dt} = C_k |V| \left( s_0^* - s_b \right) + C_d \frac{|V|^3}{T_b} \] (12)

Boundary layer entropy:
\[ h \frac{dM}{dt} = -r |V| V \] (13)

Boundary layer angular momentum:
Combine (12) and (13):

\[
\frac{ds_b}{dM} = -\frac{C_k}{C_D} \left( s_0^* - s_b \right) - \frac{|V|^2}{T_b r V}
\]

Let \( s_b \approx s^* \), \( |V| \approx V \approx V_b \), \( r \approx r_b \)

\[
\rightarrow \frac{ds^*}{dM} = -\frac{C_k}{C_D} \frac{\left( s_0^* - s^* \right)}{r_b V_b} - \frac{V_b}{T_b r_b}
\]  \( \text{(14)} \)

Balance condition (1):

\[
\frac{V_b}{r_b} = -\left( T_b - T_o \right) \frac{ds^*}{dM}
\]  \( \text{(15)} \)
Eliminate $V_b$ between (14) and (15):

$$
\left( \frac{ds^*}{dM} \right)^2 = \frac{T_b}{T_o} \frac{C_k}{C_D} \frac{s_0^* - s^*}{r_b^2 (T_b - T_o)}
$$

(16)

Eliminate $r_b^2$ between (11) and (16):

$$
\left( \frac{ds^*}{dM} \right)^2 + 2\chi \frac{ds^*}{dM} - \frac{\chi f}{T_b - T_o} = 0,
$$

(17)

where

$$
\chi \equiv \frac{T_b}{T_o} \frac{C_k}{C_D} \frac{s_0^* - s^*}{2M}
$$

Remember that

$$
\frac{\partial T_o}{\partial M} \approx - \frac{Ri_c}{r_t^2} \left( \frac{dM}{ds^*} \right)
$$

(10)
inward from some outer radius $r_o$, defined such that

$$V = 0 \quad \text{at} \quad r = r_o$$

In general, integrating this system will not yield $T_o = T_t$ at $r = r_{max}$. Iterate value of $r_t$ until this condition is met.

If $V \gg fr$, we ignore dissipative heating, and we neglect pressure dependence of $s_0^*$, then we can derive an approximate closed-form solution.
Assuming that $Ri$ is critical in the outflow leads to an equation for $T_o$ that, coupled to the interior balance equation and the slab boundary layer lead (surprisingly!) to a closed form analytic solution for the gradient wind (as represented by angular momentum, $M$, at the top of the boundary layer:

$$
\left( \frac{M}{M_m} \right)^2 \frac{C_k}{C_D} = \frac{2 \left( \frac{r}{r_m} \right)^2}{2 - \frac{C_k}{C_D} + \frac{C_k}{C_D} \left( \frac{r}{r_m} \right)^2}, \tag{18}
$$
Evaluate at \( r_o \):

\[
\left( \frac{fr_o^2}{2V_m r_m} \right)^{2-C_r \over C_D} = \frac{2 \left( \frac{r_o}{r_m} \right)^2}{2 - C_k \over C_D + C_k \left( \frac{r_o}{r_m} \right)^2}.
\]  

(19)

For \( r_o \gg r_m \):

\[
r_m \approx \frac{1}{2} fr_o^2 V_m^{-1} \left( \frac{1}{2} \frac{C_k}{C_D} \right)^{\frac{1}{2-C_r \over C_D}}.
\]  

(20)

The maximum wind speed, \( V_m \), found from maximizing the radial dependence of wind speed in the solution (18) on previous slide is given by

\[
V_m^{2-C_r \over C_D} = V_p^2 \left( \frac{2r_m}{fr_o^2} \right)^{C_k \over C_D}
\]  

(21)
Substituting (20) into (21) gives

\[ V_m^2 \equiv V_p^2 \left( \frac{1}{2} \frac{C_k}{C_D} \right)^{2-\frac{C_k}{C_D}} \]  

Can be calculated directly from SST and soundings

\[ V_p^2 \equiv \frac{C_k}{C_D} (T_b - T_t)(S_0 - S_e) \]
\[ r_m = \left( \frac{1}{2} \right)^{\frac{3}{2}} \frac{fr_o^2}{\sqrt{(T_b - T_t) (S_0^* - S_e^*)}} \]

Also,

\[ r_t^2 = r_m^2 \frac{C_D}{C_k} Ri_c \]
Numerical integrations with RE87 model (no dissipative heating, no pressure dependence of $k_0^*$): Left, regular variables; Right: Velocity scaled by (31) and time scaled by the inverse square-root of the enthalpy exchange coefficient.
Effects of Pressure-Dependence of Surface Saturation Enthalpy
Maximum Wind Speed (m/s)

$\mathcal{H} = 0.75 \quad C_k/C_D = 1.2$
Thermodynamic disequilibrium is necessary to maintain ocean heat balance:

**Ocean mixed layer Energy Balance** (neglecting lateral heat transport):

\[
C_k \rho \mid V_s \mid \left( k_0^* - k \right) = F_\downarrow - F_\uparrow - F_{\text{entrain}}
\]

\[
V_p^2 = \frac{T_s - T_o}{T_o} \frac{F_\downarrow - F_\uparrow - F_{\text{entrain}}}{C_D \rho \mid V_s \mid}
\]

Greenhouse effect decreases this

Weak explicit dependence on \( T_s \)

Mean surface wind speed

Ocean mixed layer entrainment
Dependence on Sea Surface Temperature (SST):

- Mean Slope = 8 m s\(^{-1}\) K\(^{-1}\) when \(U_{\text{sfc}}\) varied
- Average dependence when radiation varied: 3 m s\(^{-1}\) K\(^{-1}\)
Relationship between potential intensity (PI) and intensity of real tropical cyclones

The graph shows the cumulative frequency of normalized wind speeds for hurricanes and typhoons, along with tropical storms. The data points represent different wind speeds, with hurricanes and typhoons indicated by red circles and tropical storms by blue diamonds. The trend lines suggest a decreasing frequency as wind speeds increase.